

Temporal complexity of the order parameter at the phase transition

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We study a decision making model in a condition where it is equivalent to the two-dimensional Ising model, and we show that at the onset of phase transition it generates temporal complexity, namely, nonstationary and nonergodic fluctuations. We argue that this is a general property of criticality, thereby opening the door to the application of the recently discovered phenomenon of complexity matching: For an efficient transfer of information to occur, a perturbing complex network must share the same temporal complexity as the perturbed complex network.

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I. INTRODUCTION

Phase transitions and critical phenomena occur frequently in nature and have been widely studied by physicists, see for instance [1]. The Ising model [2] originally introduced to explain ferromagnetic phase transition is well known, and the exact solution found by Onsager [3] for the occurrence of phase transition in the two-dimensional case is widely recognized as an example of outstanding theoretical achievement. In the last few years some scientists have used the Ising model to shed light on biological and neurophysiological processes [4–7]. More precisely, the authors of [4] used the Ising model to explain the collective behavior of biological networks and the authors of [5–7] adopted the Ising model for the purpose of supporting their hypothesis that the brain works at criticality without establishing a clear distinction between phase transition and self-organized criticality [8]. Finally, we have to mention that the Ising model is frequently used, see for instance [9,10], to model neurophysiological data with the constraint of maximal entropy. Although relevant to explore the cognition properties of the brain, this is a perspective different from the one adopted in this paper.

Couzin [11] made the conjecture that the cooperative interaction between the birds of a flock may mimic the brain cognition properties, and the experimental observation of Cavagna *et al.* [12] confirmed the criticality condition of a coordinated flock of birds. It is important to emphasize that the connection between phase transition processes and cooperative behavior of biological systems was already stressed in the pioneering paper of Vicsek *et al.* [13]. It is thought [4] that biological systems in general operate at criticality.

The phase transition condition is not confined to biological systems. The recent work on econophysics [14,15] suggests that the same arguments can be applied to sociological systems. As a matter of fact, Sornette [15] proposes a model of cooperative economical interaction, of the same type as the Ising model, which generates at criticality a transition from subcritical to supercritical patterns very similar to those produced by the Ising model. As stressed by the authors of [5], the Ising patterns at criticality correspond to the emergence of correlation links yielding a scale-free network statistically indistinguishable from that experimentally observed within the brain, using functional magnetic resonance imaging.

These studies emphasize the spatial and network complexity emerging from the cooperative interaction of the network's units, but overlook the temporal complexity of these networks. Herein we are attempting to fill this gap and prove that the temporal complexity emerges at criticality. Temporal complexity is defined as follows. The time dynamics of complex networks is characterized by the occurrence of significant events, which may be financial crashes [15], brain quakes [5,6], or the changes of direction of a flock of birds [12,13]. The time interval between two consecutive events is given by a distribution density $\psi(t)$, which, drastically departing from the conventional Poisson statistics, has the inverse-power-law form

$$\psi(\tau) \propto \frac{1}{\tau^\mu}, \quad (1)$$

with $\mu < 2$. The occurrence of an event does not have any memory of the occurrence of earlier events. This property is usually denoted as renewal, but it must not be confused with the ordinary Poisson and Markov condition: The signal generated by these events is characterized by long-range correlation in time, and, most importantly, it is essentially nonstationary, thereby breaking the ergodicity that is a fundamental property of statistical physics [16]. We refer to it as the non-Poisson renewal condition. A significant event is interpreted as a failure whose occurrence brings the network back to a brand new condition. We refer to these as *crucial events*.

Events of this kind may be very difficult to detect. The theoretical discussion [16], for instance, applies to the case of dichotomous signals, where crucial events correspond to the abrupt transition from one to the other value of the signal. In the case of the decision making model (DMM) under study, the single units in isolation produce perfect dichotomous signals. They can be easily detected, but they are of a Poisson kind, thereby strikingly departing from the condition of temporal complexity. When the control parameter is extremely large, the global signal undergoes abrupt transitions that however depart again from the condition of temporal complexity, because they are predominantly of a Poisson kind. At criticality the global signal loses these abrupt transitions that are replaced by smoother fluctuations around the mean value. Herein we consider a model generating a signal whose time mean value vanishes. We assume the crossing of the origin to be the proxy of a significant event. Thus, we interpret the signal as a

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dichotomous fluctuation making abrupt changes from 1 to -1 and back again. We evaluate the correlation function of this ideal dichotomous signal, denoted by the symbol $\Psi(t, t')$, and we prove that its dependence on the times t and t' exactly fits the prediction of the non-Poisson renewal condition.

The connection between phase transition, and its spatial and network complexity, with temporal complexity is not well known. In the literature there are only a few papers where this connection is discussed. One is [17], which converts the 3D Ising model into a critical map generating type I intermittent dynamics and consequently the same temporal complexity as the one under discussion herein. The emergence of non-Poisson renewal properties was also discussed in [18,19], which is devoted to illustrating the cooperative properties of a DMM. The foundation of these papers is sociological and similar to that of the economical model illustrated in the work of Sornette [15]. Thus, the authors [18,19] did not discuss the connection of the DMM with the Ising model and adopted the nonrealistic but often made assumption of all-to-all coupling among the network's units, thereby confining the emergence of an inverse power law, namely, of temporal complexity, to the same time scale as that of the units in isolation.

The motivation for the present research work is given by the transfer of information from a complex network to another complex network [20]. Rather than using entropic arguments, as is usually done with the transfer of information, we open the door to the concept of complexity management recently advocated in [21]. This latter work established that a complex network driven by crucial events is sensitive to stimuli with the same complexity. In other words, the transfer of information rests on temporal complexity and this explains the motivation to prove that criticality also generates temporal complexity.

In Sec. II we discuss under which conditions the DMM [18, 19] is equivalent to the Ising model. In Sec. III we prove that at criticality the function $\psi(t)$ becomes an inverse power law for over four decades. We shall also argue that with increasing the network's size $\psi(t)$ may become an infinitely extended inverse power law. Section IV shows that this is a renewal process. Finally we devote Sec. V to concluding remarks.

II. DETAILED DESCRIPTION OF THE MODEL

We consider a system of L discrete variables located at the nodes of a two-dimensional square lattice. Each unit s_i is a stochastic oscillator and can be found in either of two states, $+1$ or -1 . The dynamic is introduced by choosing a single unit on site i and updating it in an elementary time step with a transition rate g :

$$g(s_i^{+1} \rightarrow s_i^{-1}) = g_0 \exp \left[\frac{K}{M} (M_{+1} - M_{-1}) \right], \quad (2)$$

$$g(s_i^{-1} \rightarrow s_i^{+1}) = g_0 \exp \left[-\frac{K}{M} (M_{+1} - M_{-1}) \right]. \quad (3)$$

Here M denotes the total number of nearest neighbors, and M_{+1} and M_{-1} the number of nearest neighbors being in the state $+1$ and -1 , respectively. Single units change its states, thereby making M_{+1} and M_{-1} fluctuate in time, while, of course, the total number of nearest neighbors is conserved, $M_{+1} + M_{-1} = M$.

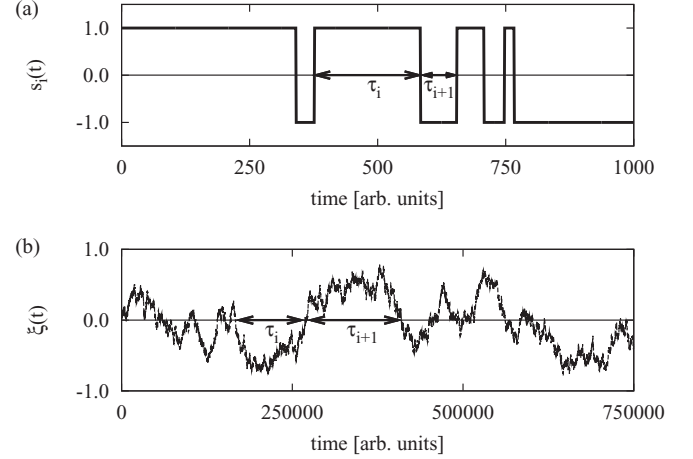


FIG. 1. (a) Temporal evolution of a single unit $s_i(t)$ and (b) of the global order parameter $\xi(t)$ for the decision making model realized on a square lattice of $L = 50 \times 50$ nodes, with $g_0 = 0.01$ and $K = 1.70$. To illustrate the concept of crucial events we mark the time intervals τ between two consecutive events, according to their definitions assumed in this paper. Notice different time scales on both plots.

All numerical calculation are performed on a square lattice of either $L = 50 \times 50$ or $L = 100 \times 100$ nodes with periodic boundary conditions. In a single time step a run over the whole lattice is performed and for every unit s_i the transition rate of Eqs. (2) or (3) is calculated according to which a node is given the possibility to change its state. The single unit in isolation, $K = 0$, fluctuates between states $+1$ and -1 with the transition rate $g = g_0$. When coupling constant $K > 0$, a unit in the state $+1$ (-1) makes a transition to the state -1 ($+1$) faster or slower according to whether $M_{-1} > M_{+1}$ ($M_{+1} > M_{-1}$) or $M_{-1} < M_{+1}$ ($M_{+1} < M_{-1}$), respectively.

Next, we define the global order parameter $\xi(t) = \frac{1}{L} \sum_{i=1}^L s_i(t)$, which is characterized by the variability that does not possess the dichotomous character of single units. In Fig. 1 we show exemplary temporal evolution for the single unit $s_i(t)$ and for the global order parameter $\xi(t)$. Note that the amplitude of variable $\xi(t)$ depends on the value of the coupling constant K . When $K = 0$, all units in the system are independent Poisson processes; thereby an average taken at any moment of time over all of them is zero. Once the value of the coupling increases, $K > 0$, single units are less and less independent, resulting in a nonzero average. The quantity K_C is the critical value of the control parameter K , at which point a phase transition to a global majority state occurs. In numerical calculations we use the time average $\xi_{eq} = \langle |\xi(t)| \rangle$ as a measure of this global majority. More precisely, after initial 10^6 time steps, the average is taken over the same number of the consecutive time steps of the model.

Note that in the special case when M is the same for all the nodes and $g_0 \ll 1$, we find that our model generates the same phase transition as the two-dimensional Ising model discussed in the seminal paper of Onsager [3]. This is an expected result insofar as the Ising model rests on the Hamiltonian

$$H = -J \sum_i^L \sigma_i \sigma_j, \quad (4)$$

where σ_i and σ_j denote the Pauli operators with eigenvalues ± 1 . According to Boltzmann statistics equilibrium is described by the density matrix

$$\rho = \frac{\exp(-\beta H)}{Z}, \quad (5)$$

with $\beta = \frac{1}{k_B T}$, k_B being the Boltzmann constant and T the absolute temperature. Thus, the off-diagonal elements of the transfer matrix [22] become equivalent to the transition rates of Eqs. (2) and (3), under the condition

$$k_B T = \frac{J}{K} = \frac{M}{K}. \quad (6)$$

As examples of conditions yielding this equivalence, we consider two cases. The former is the case of all-to-all coupling, where $M = L$. The latter is considered here, and it is the two-dimensional lattice where each node is coupled to its 4 nearest neighbors, thereby setting $M = 4$.

The thermodynamical condition $M = L = \infty$ was discussed extensively by authors of [18,19], who showed that under those conditions the ratios $\frac{M_{\pm 1}}{M}$ are equivalent to the probabilities $p_{\pm 1}$ for a node s_i to be in one of two allowed states. The dynamic evolution of a single unit state is then described by a two-state master equation

$$\frac{d}{dt} p_{+1} = -g(s_i^{+1} \rightarrow s_i^{-1}) p_{+1} + g(s_i^{-1} \rightarrow s_i^{+1}) p_{-1}, \quad (7)$$

$$\frac{d}{dt} p_{-1} = -g(s_i^{-1} \rightarrow s_i^{+1}) p_{-1} + g(s_i^{+1} \rightarrow s_i^{-1}) p_{+1}. \quad (8)$$

Solving the above for the difference in probabilities $\Pi(t) = p_{+1}(t) - p_{-1}(t)$, which corresponds to the earlier defined global order parameter $\xi(t)$, one obtains

$$\frac{d}{dt} \Pi = 2g_0 [\sinh(K \Pi) - \Pi \cosh(K \Pi)]. \quad (9)$$

This equation yields two solutions, corresponding to global majority states, for the values of coupling constant $K > K_C$, where $K_C = 1$.

The solution to the latter condition of $M = 4$, $L = \infty$ can be found in [23] and yields the condition for the global variable to be $\xi_{eq} = (1 - [\sinh(K/2)]^{-4})^{1/8}$. In this case the critical value of the coupling constant is $K_C = 2 \ln(1 + \sqrt{2}) = 1.7627$. In Fig. 2, we show the phase diagram in the latter condition and we find that, as expected, the numerical evaluation of $\xi_{eq}(K)$ is very close to the theoretical prediction of Onsager [3], thereby confirming that the DMM is equivalent to the Ising model in the limiting case $g_0 \rightarrow 0$.

This equivalence between the DMM and the Ising model is merely formal, because the DMM does not have a Hamiltonian origin and does not require the action of a thermal bath at temperature T to work as does the Boltzmann picture. This explains why the equivalence with the Ising model requires that g_0 vanish, so as to freeze the dynamics of the single units, in the absence of cooperation.

When we release the condition $g_0 \rightarrow 0$, the equivalence of the DMM with the Ising model is lost. In Fig. 3, we see that, if the condition of $g_0 \ll 1$ is abolished, the phase transition emerges at values of K lower than K_C of the theoretical prediction. Simultaneously, further increase of g_0 ($g_0 \approx 0.40$) leads to a new regime, in which every unit is surrounded by

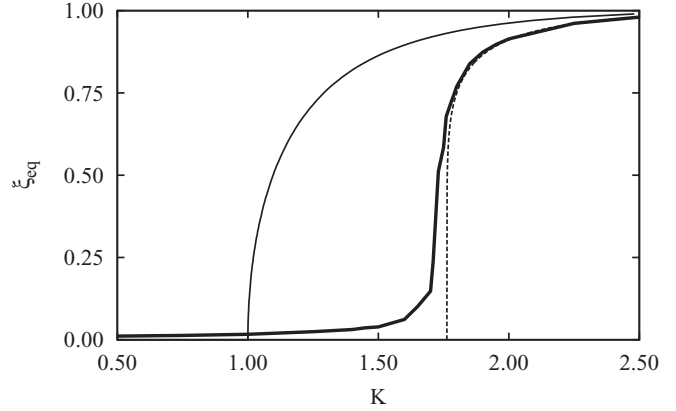


FIG. 2. The phase diagram for global variable ξ_{eq} . Thin solid line and dashed line are the theoretical predictions for the fully connected and two-dimensional regular network, respectively. In both cases $L = \infty$ and the latter case is the Onsager theoretical prediction [3] for 2D regular lattice. The thick solid line corresponds to the global states observed for two-dimensional regular lattice ($L = 100 \times 100$ nodes) and $g_0 = 0.01$. Periodic boundary conditions were applied.

nearest neighbors in the opposite state, yielding an update of its state at every time step and generating the condition in which the order parameter $\xi(t)$ is exactly zero at all times. It is worth pointing out that when a network characterized by a large transition rate g (transition rate $g \approx 1$) is studied, it shows sensitivity to the initial random configuration of the lattice. We observed that in a limited number of cases the order parameter evolves to a global majority state, in which all the nodes are in the same state rather than in condition $\xi(t) = \xi_{eq} = 0$. Therefore one may consider a bifurcation in the phase space of the model parameters in this regime.

III. ORDER PARAMETER REVERSAL TIMES

Let us now study the temporal complexity of the order parameter $\xi(t)$. As pointed out in Sec. I, to prove temporal complexity of $\xi(t)$ we have to observe significant events and prove that they are crucial. At criticality the signal is not dichotomous and, as we shall see hereby, there are good reasons why it *must depart from the dichotomous condition*.

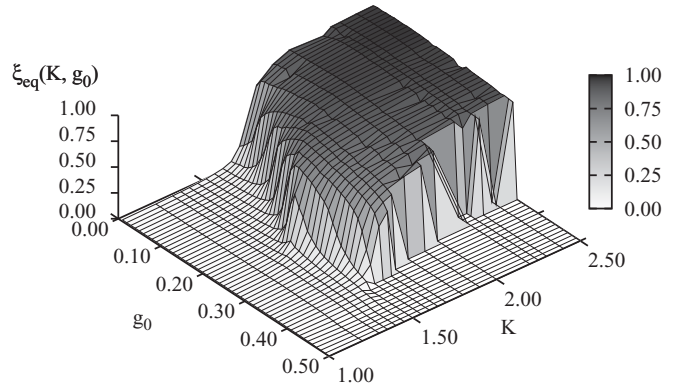


FIG. 3. The two-dimensional phase diagram of global variable ξ_{eq} evaluated for a range of model parameters, g_0 and K , on a two-dimensional regular lattice of size $L = 100 \times 100$ nodes.

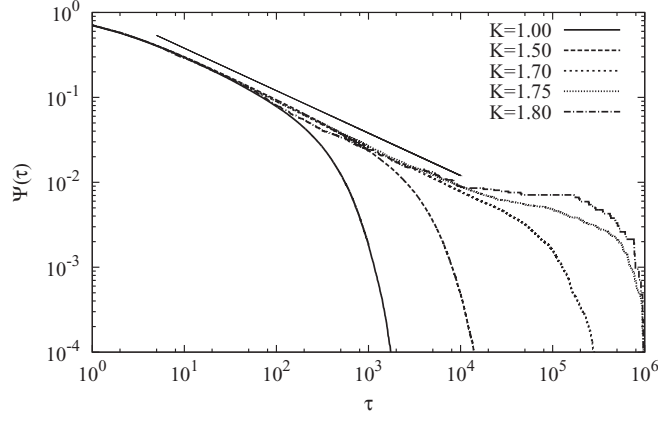


FIG. 4. Survival probability function $\Psi(\tau)$ for the order parameter $\xi(t)$ evaluated on a two-dimensional lattice of size $L = 50 \times 50$ for $g_0 = 0.01$ and increasing values of coupling constant K . The straight line corresponds to the slope of -0.50 , namely to $\mu = 1.50$, since $\Psi(\tau) \sim \frac{1}{\tau^{\mu-1}}$.

We make the conjecture that the crossing of the origin, namely the times at which $\xi(t)$ changes sign, are the significant events to observe. As illustrated on Fig. 1, we interpret the time interval τ between two consecutive crossings as the time duration of a given decision, even if this decision may rest on a slight and fluctuating majority.

We evaluate the distribution density of decision-time durations τ , $\psi(\tau)$, and the corresponding survival probability $\Psi(\tau)$, where $\Psi(t) = \int_t^\infty d\tau \psi(\tau)$. Although emerging from a simple regular lattice, that is, one with no structural complexity, the survival probability presented in Fig. 4 shows a scale-free property that extends over more than four decades in time for $K \approx K_C$. A further increase of K does not affect the power-law region and has the effect of producing a more and more extended exponential shoulder. The exponential shoulder is expected to become predominant for $K \rightarrow \infty$.

As mentioned earlier, a single unit in isolation fluctuates between two states with the transition rate $g = g_0$. The corresponding survival probability function is an exponential function $\Psi(\tau) = \exp(-g_0\tau)$. At the same time, as illustrated on Fig. 5, a coupled unit tends to update its state with a transition rate smaller than g_0 . This is a property of criticality that is lost completely for very large values of the control parameter K . Although it is a computational challenge to explore the dynamics corresponding to extremely large values of K , our numerical results suggest that when K is very close to the critical value, and a decision is reached by a slight majority, the single units have a dynamics almost indistinguishable from the Poisson dynamic that they would have in isolation. As we increase the control parameter K and the majority becomes larger the single units keep their Poisson dynamics with a smaller rate. Although it is not possible because of computer-time limitations to study the network's dynamics for values of K much larger than $K = 1.80$, it is plausible to make the conjecture that the single units maintain their Poisson dynamics and that these dynamics become closer and closer to that of the global variable that will lose its power-law structure and will become predominantly exponential. In other words, unanimous and permanent consensus must be perceived as the

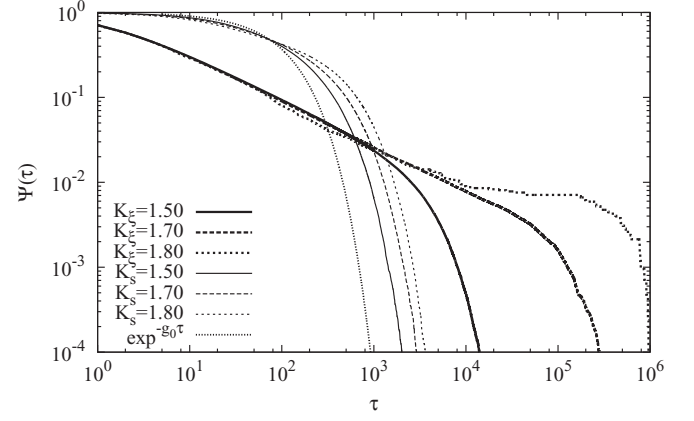


FIG. 5. The survival probability function $\Psi(\tau)$ of the global order parameter $\xi(t)$ is compared with the transitions between two states for a single unit s_i . Simulations were performed on a lattice of size $L = 50 \times 50$ for $g_0 = 0.01$ and increasing values of coupling constant K .

manifestation of a limiting condition of Poisson but infinitely slow dynamics, where the behavior of the single units becomes identical to that of the global signal.

To make this argument more compelling let us discuss the properties of the observed exponential shoulder. Since one observes the network in an organized phase ($K > K_C$) it is not surprising that one perceives the exponential signature of an equilibrium regime. Therefore we assume that under those conditions the global variable $\xi(t)$ follows the dynamics of a particle diffusing in a double potential well. In the equilibrium condition one would expect the survival time of staying in one well to be $\Psi(\tau) = \exp(-a\tau)$, where $a = A \exp(-Q/D)$. The parameter Q denotes the height of the potential barrier separating the wells, and D stands for the diffusion coefficient. Following [19] we expect the barrier to be a function of the coupling constant K and the diffusion to depend on the number of nodes L . By fitting the shoulder, once for a case where L is kept constant and K varies, and secondly for an opposite condition, we assess the above hypothesis. The fitting procedure revealed that the barrier height Q is a linear function of K ($Q \sim K$) and that the diffusion coefficient D is inversely proportional to the system size L ($D \sim 1/L$). For $L \rightarrow \infty$ the transition rate becomes infinitely small and the exponential shoulder becomes predominant and virtually coincident with the Poisson dynamics of the single units.

While in the organized phase the effect of increasing L is that of making the process exponential, although with a virtually infinite transition time. At criticality the increase of L has a dramatically different effect. The authors of [19], based on this DMM in the all-to-all condition, found that at criticality an inverse-power-law behavior emerges, which is confined, however, to a time region with the same size as the Poisson time $\tau_P \approx 1/g_0$. However, this is not the case when the units are the nodes of a regular two-dimensional lattice, interacting only with the four nearest-neighbor nodes. We have evaluated the time size of the inverse-power-law region appearing before the exponential shoulder and we found that its size tends to increase with increasing L as $\tau_P \sim L^{-1.25}$. Unfortunately, this observation is limited to values of L

smaller than $L = 10\,000$, since going beyond would exceed the limits of our computer facilities. However, on the basis of this observation we reach the conclusion that for $L \rightarrow \infty$ the size of the temporal complexity must become infinitely large. Thus, criticality is a kind of complex singularity embedded in a Poisson sea.

These intuitive arguments also explain why the signal $\xi(t)$ at criticality must depart from the dichotomous condition. This is a consequence of the fact that the single units are not rigidly bound to follow the opinion of the majority. There is, consequently, a subtle connection between criticality, free will, and correlation between different units. Although the single units may change opinion, they cannot do that in a way totally uncorrelated from the behavior of the other units, insofar as this would be incompatible with the emergence of a majority, as slight as it might be.

IV. TESTING RENEWAL PROPERTIES

As pointed out earlier, the signal $\xi(t)$ at criticality is not dichotomous. However, replacing it with a dichotomous signal corresponding to 1 (yes) when $\xi(t) > 0$ and to -1 (no) when $\xi(t) < 0$, thereby turning it into an ideal dichotomous signal, is the most convenient way to reveal the emergence of crucial events at criticality. The theory [24] (see also the earlier work [25]) allows us to evaluate the nonstationary correlation function $\Psi(t, t')$ in the specific case when the renewal process rests on the waiting time distribution density $\psi(t)$ assumed to be an ideal inverse power law. In general, regardless of whether this ideal condition of infinitely extended inverse power law is realized or not, it is possible to establish whether the process is renewal by noticing that

$$\Psi(t, t') = \Psi(\tau, t_a), \quad (10)$$

where $\tau = t - t'$ and $t_a = t'$. As explained [26], the function $\Psi(\tau, t_a)$ can be evaluated by using the time series $\{t_i\}$, where t_i are the times at which the fluctuation $\xi(t)$ crosses the origin. We use a mobile window of length t_a to evaluate the waiting time distribution $\psi(\tau, t_a)$ and the corresponding survival probability $\Psi(\tau, t_a)$, by locating the origin of the window on the time of occurrence of an event and measuring the time distance between the end of the window and the time of occurrence of the first event after the window end. To establish whether the process is renewal or not we shuffle the sequence of reversal times, thereby generating a new time series $\{t_i^s\}$. We use the same procedure based on a mobile window to determine $\Psi(t_{a,s}, \tau)$, which corresponds to the nonstationary correlation function in the renewal case. If the two survival probabilities coincide, we conclude that the process is renewal. If the renewal process is exponential, the correlation function is stationary, and there is no aging.

In the ideal case of an infinitely extended inverse power law, with power index $\mu < 2$, as discussed in earlier work [16,24], aging is perennial. In the case studied in this paper, we have to take into account that this ideal condition would be realized by assigning an infinitely large value to L , with the ensuing consequence that temporal complexity would become virtually invisible, due to the joint effect of an extremely slight majority and of a very extended sojourn in a given decision state. To make temporal complexity visible, we need a compromise,

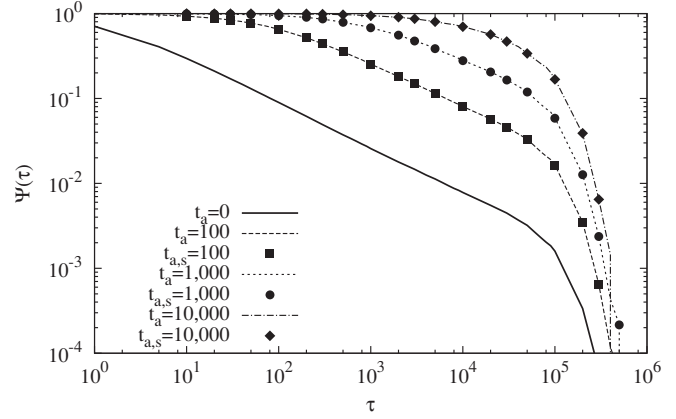


FIG. 6. Testing for renewal property. Survival probability function $\Psi(\tau)$ of the order parameter $\xi(t)$ evaluated on lattice of size $L = 50 \times 50$ and $g_0 = 0.01$, $K = 1.70$ was aged respectively by time $t_a = 100$, $t_a = 1000$, and $t_a = 10\,000$. Those aged survival probabilities $\Psi(t_a, \tau)$ are compared with the aged renewal prediction $\Psi(t_{a,s}, \tau)$.

and we have to set a limit on the time extension of the inverse power law. As a consequence, we obtain the very interesting result depicted in Fig. 6, where aging, namely the slowing down of the survival probability, becomes very large in the inverse-power-law region, with the survival probability remaining virtually constant, and in the correspondence of the exponential shoulder it is virtually suppressed, in accordance with the principle that Poisson statistics annihilates aging.

V. CONCLUSIONS

This paper establishes that at the onset of phase transition, in addition to the spatial and network complexity shown by a number of earlier investigations [4–7,9,12,14,15], we also have temporal complexity. This significant result reveals a path for the transport of information from one complex network to another. As pointed out in Sec. I, there is a general agreement that complex systems are a set of many units interacting at the onset of phase transition. The present analysis proves that these complex networks are characterized also by temporal complexity, and consequently a perturbing complex network is expected to exert its influence on another complex network via the recently discovered complexity management process [21,31]. Although the present theoretical predictions are based on the assumption that $\psi(\tau)$ is an ideal inverse power law and the real complex networks, as shown herein, reach this ideal condition only when their size is infinitely large, the result in Fig. 5 indicates that the time region generating ergodicity breakdown may become so extended as to make a complex network virtually insensitive to stimuli that do not share the same extended nonergodic condition.

In the last few years the attention of investigators has been moving from chaos synchronization of two nonlinear oscillators [27] to the synchronization dynamics of many units in large-scale networks [28], called *inner synchronization* [29] for convenience. The cooperation induced phase transition discussed herein can to some extent be thought of as a form of

inner synchronization. As pointed out [29] the challenge that the researchers in the field of complexity have now to address is the phenomenon of *outer synchronization* [30].

In this light, we conclude that the results presented here provide a way of addressing outer synchronization, redirecting the attention of researchers from the details of the topology of the complex networks to the analysis of the temporal complexity that inner synchronization may generate. In fact the

principle of complex management [21,31] rests on temporal complexity, regardless of the way it is generated.

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